Anisotropic critical properties of the hexatic $N+6$ phase near the transition to the hexagonal discotic phase

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We treat the theoretically predicted hexatic $N+6$ phase near the transition to the hexagonal discotic phase by means of a renormalization technique. In particular, we investigate one-loop corrections to the order-parameter correlation function. The condensation of the hexagonal two-dimensional lattice of the ordered phase from the bond-orientationally-ordered $N+6$ phase was described in a previous work [C. Giannessi, Phys. Rev. A **28**, 350 (1983)] using a triple mass-density wave as an order parameter. The requirement of local invariance under rotations gives rise to gauge coupling between the order parameter and the bond-angle field Ω, introduced to fix the full orientational order of the hexatic phase. The peculiar tensorial structure of such a gauge coupling yields anisotropic corrections to the order-parameter correlation function, even in the plane orthogonal to the nematic director with respect to the directions of the sixfold crystallographic axes. As a consequence, the structure factor for x-ray diffraction should acquire sixfold anisotropy, revealing anisotropic scaling of correlation lengths. Actually, the occurrence of anisotropic scaling depends on the existence of a stable anisotropic fixed point of renormalization flow. For technical reasons it has not been possible to determine and analyze the fixed point controlling the phase transition, so that we can only state the possibility of anisotropic critical behavior characterized by correlation lengths in directions parallel and orthogonal both to the director and to the sixfold crystallographic axes diverging with different critical exponents.

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I. INTRODUCTION

Bond-orientationally-ordered phases [1–3], as intermediates between a fully disordered phase and a phase that is both orientationally and translationally ordered, should be a universal feature of ordered media. Such new phases have been theoretically predicted, on symmetry grounds, in a wide class of systems, such as Lifshitz point systems and various kinds of liquid crystals, which share some symmetry features despite different physical structures [3]. On the other hand, for most of these phases there is not yet experimental evidence. In fact, the actual range of stability of a particular intermediate phase could depend on fluctuation-induced breaking of long-range orientational order, which might make experimental observation of these phases very hard.

In discotic liquid crystals, according to our model for the melting of the hexagonal discotic phase into the nematic phase [4,5], we predicted the existence of an intermediate hexatic $N+6$ phase [2,4], which is still experimentally undiscovered. As regards symmetry, the hexatic phase is intermediate between the disordered phase and the ordered one: it is translationally invariant and therefore has homogeneous density like the nematic phase, while exhibiting long-range sixfold orientational order around the director like the hexagonal discotic phase.

In fact, the hexagonal discotic phase can be viewed as a quasi-two-dimensional system [4], consisting of an array of liquid columns parallel to each other, the axes of which are regularly positioned on a two-dimensional hexagonal lattice. The liquidlike behavior along the columns should stabilize the two-dimensional lattice against thermal fluctuations [4], so making the system stable in the thermodynamic limit, while the strictly two-dimensional lattice is weakly unstable when the size of the system goes to infinity [6]. In any other regard, the system is analogous with a two-dimensional lattice and therefore the phase diagram should present an intermediate phase, in agreement with the Nelson-Halperin theory of two-dimensional melting [7]. In this connection, the stability of the $N+6$ phase against fluctuation-induced breaking of long-range orientational order has just been tested [8].

As an order parameter for the condensation of the hexagonal phase from the intermediate $N+6$ phase, we used a set of complex parameters describing the onset of a triple mass-density wave in the plane orthogonal to the nematic director (conventionally taken along the $\hat{\mathbf{z}}$ axis) [4]. The hexatic phase, on the contrary, is melted as regards the two-dimensional lattice, but it maintains a vestige of the singled-out crystallographic axes in the hexagonal anisotropy. Therefore, the nematic director is no longer sufficient to fix orientational order. That is why a bond-angle field Ω_z was introduced, in order to give the local orientation of the two-dimensional lattice in the XY plane. Such Ω_z was defined as the rotation angle around the unperturbed nematic director $\mathbf{m}_0$ between a given reciprocal lattice vector and a fixed $\hat{\mathbf{x}}$ axis. Even if in the hexatic phase the two-dimensional lattice is melted, the "directions" of the reciprocal lattice vectors, i.e., the crystallographic axes, remain. The full orientational order of the hexatic phase is therefore fixed by the local ro-

tation field $\mathbf{\Omega}$ [4]:

$$\mathbf{\Omega}=\Omega_z\mathbf{m}_0+(\mathbf{m}_0\times\delta\mathbf{m})\ , \tag{1.1}$$

where $\delta\mathbf{m}$ is a small distortion of $\mathbf{m}_0$. The field $\mathbf{\Omega}$ gives the local orientation of the hexatic phase with respect to the uniform configuration where Ω_z and $\delta\mathbf{m}$ vanish.

The requirement of local invariance under rotations [1,4] yields the coupling between the aforementioned order parameter and the local field $\mathbf{\Omega}$, which can be considered as a gauge field. In fact, as for smectics [9], free energy must be invariant under global and local rotations of the system, in this case under simultaneous rotations of the liquid columns and of the director, as well as under simultaneous torsions of the columns and Ω_z rotations of the two-dimensional lattice. The full elastic energy associated with the strains of $\mathbf{\Omega}$ was calculated in [5].

In the present paper we will go beyond the mean-field approximation developed in [4] and [5], and treat the hexatic $N+6$ phase near the transition to the hexagonal phase by means of a renormalization technique. In particular, we consider the gauge coupling between the order parameter, representing a mass-density wave, and the curvature field $\mathbf{\Omega}$, which is the gauge field. Such an interaction yields one-loop corrections to the order-parameter propagator and then renormalizes the density-density correlation function. The Fourier transform of the density-density correlation function represents the structure factor for the x-ray diffraction, and therefore physical effects on its form are experimentally observable.

As a physical picture, the propagating density wave is distorted because the matter is coupled to the curvature gauge field, so that fluctuations of orientational order drive critical behavior of mass-density space modulation at different wave vectors. The local distortion of the orientational configuration influences the space propagation of density waves.

The calculation of such a renormalization effect can be carried out by Feynman diagrams, as usual in field theory, where they represent the "radiative" corrections to the classical fields. The actual computation of such diagrams requires the knowledge of the propagators of the gauge fields, which in our case are the director elastic modes $\delta\mathbf{m}$ and the "rotation" elastic modes Ω_z. The full elastic energy associated with the strains of $\mathbf{\Omega}$ [5] supplies the free propagators of the gauge fields, i.e., the propagators of the pure gauge fields, not coupled to the matter. The Frank elastic constant γ_3, which couples $\delta\mathbf{m}$ to Ω_z [5], shows critical enhancement [8], so that the director elastic modes $\delta\mathbf{m}$ remain strongly coupled to the "rotation" elastic modes Ω_z, near the transition temperature too. Such a feature implies that the normal elastic modes are a mixing of $\delta\mathbf{m}$ and Ω_z, and the corresponding inverse propagators are not simply quadratic in the wave vector [2,8].

The main result of this analysis is the possibility of anisotropic scaling of correlation lengths, in the hexatic phase, both along the $\hat{\mathbf{z}}$ axis of the unperturbed director with respect to the XY plane, and in the actual XY plane, between lengths parallel and orthogonal to the crystallographic axes, respectively. Anisotropic scaling is a consequence of the coupling between the order parameter and the gauge field $\mathbf{\Omega}$, by the one-loop renormalization corrections to the density-density correlation function. Even if the bare correlation function is isotropic in the plane orthogonal to the director [4], the gauge interaction renormalizes it with an anisotropic contribution. The anisotropic critical behavior of the renormalized correlation function is due to the tensorial structure of the gauge coupling in our model [4], which distinguishes the directions along the axes of the hexagonal two-dimensional lattice from the respective orthogonal directions, in the XY plane. As a consequence, the density-density correlation function acquires sixfold anisotropy in the XY plane (see also [1,2]).

The actual occurrence of anisotropic scaling depends on the behavior of correlation lengths at the critical fixed point of renormalization-group flow. Unfortunately, the above-mentioned complications about the inverse propagators of the gauge fields $\delta\mathbf{m}$ and Ω_z, which are not quadratic in the wave vector, prevent us from finding the fixed point of the hexatic-to-hexagonal discotic phase transition, owing to very cumbersome calculation. Therefore, we cannot state the existence of a stable anisotropic fixed point, but we can only conclude that the transition might show anisotropic scaling properties, i.e., different critical indices for correlation lengths along different directions.

In this respect, it is helpful to make a warning comparison with smectics, which show some analogies with discotics [4,8]. Anisotropic critical properties of the De Gennes model for the nematic-to-smectic-A phase transition were investigated by Lubensky and Chen [10], using the ϵ expansion in the vicinity of four dimensions and adopting an order parameter with $n/2$ complex components. As a result, for a physical value of n, i.e., $n=2$, the scaling should be isotropic, because of instability of the anisotropic fixed point. Some doubts could be raised about the result in three dimensions, obtained in the framework of ϵ expansion near four dimensions. Nevertheless, other theoretical approaches [11,12] give isotropic scaling for smectics as well. The only arguments that should lead to anisotropic scaling for smectics are based on defect theory of the smectic-A-to-nematic transition [1], and are considered untrustworthy. Following the analogy with the anisotropic critical properties of smectics [10], we could assume that the system undergoes a series of crossovers from anisotropic quasicritical behavior to isotropic quasicritical behavior and finally, for temperature near the transition point, to a first-order transition. Anyhow, such a result should be assumed with some caution for discotics, since the aforementioned analogy, as regards dimensional regularization, could be misleading (see Sec. V).

The anisotropic critical properties of the density-density correlation function are not in contrast with the elastic isotropy of the hexagonal two-dimensional lattice. In fact, the hexagonal symmetry of the discotic phase implies cylindrical symmetry for the Frank elastic energy [5,8], as well as for the ordinary elastic energy of the lattice, while the local correlation functions are not forced to be isotropic. We shall explicitly show that the elastic isotropy of the lattice is not violated, even assuming, in

the plane orthogonal to the director, two different gradient terms that control the fluctuations of the order parameter along the crystallographic axes and orthogonally to them, respectively. Anyhow, the one-loop renormalization corrections give rise to such an anisotropy in the gradient term, which cannot therefore be in contradiction with the symmetry of the model, since the renormalization procedure preserves the symmetry of the system [13], while it changes the length scale and performs "coarse graining" of the microscopic interactions.

The elastic energy of the two-dimensional hexagonal lattice, in the condensed translationally ordered discotic phase, is derived from a gradient term that shows anisotropy in the plane orthogonal to the nematic director. Such an elastic energy just turns out to be isotropic despite the anisotropic gradient term. Besides the ordinary Lamé coefficients [14] of two-dimensional elasticity, expressing the resistance to shear and compression, we also get a torsion modulus, representing the stiffness of the liquid columns against torsion and therefore peculiar to the hexagonal discotic phase as a quasi-two-dimensional system.

In Sec. II, we get the lattice elastic energy for a discotic liquid crystal. In Sec. III, we draw the free propagators of the gauge fields from the full Frank energy of discotics. In Sec. IV, we compute the renormalization effect on the order-parameter correlation function and define anisotropic critical indices for correlation lengths in the hexatic phase near the phase transition to the hexagonal discotic phase. In Sec. V, we discuss our results and further developments. Finally, in the Appendix, some calculational details are given.

II. THE LATTICE ELASTIC ENERGY FOR DISCOTICS

According to our model [4], the order parameter for the condensation of the two-dimensional hexagonal lattice from the intermediate $N+6$ phase is

$$\delta\rho(\mathbf{r})=\sum_{i=1}^{3}\eta_i(\mathbf{r})\exp(i\mathbf{q}_i\cdot\mathbf{r})+\text{c.c.}\ , \tag{2.1}$$

which describes the onset of a triple mass-density wave in the plane orthogonal to the nematic director, with complex local amplitudes η_i $(i=1,2,3)$ and shortest reciprocal lattice vectors $\mathbf{q}_i$. The local fluctuations of the order parameters η_i are assumed to vary slowly over typical lattice spacing.

The sixfold symmetry of the two-dimensional hexagonal lattice brings forth elastic isotropy, and then the lattice elastic energy must be invariant under rotations in the plane of the lattice (XY plane). Such an overall isotropy of elastic and optical properties is due to a macroscopic average over local interactions. On the contrary, the local correlation functions, which describe the response of the system to short-wavelength perturbations, are sensitive to the symmetry of the interactions on a microscopic scale. Therefore, the density-density correlation function and then the structure factor for the x-ray diffraction $S(\mathbf{q})=\langle|\delta\rho(\mathbf{q})|^2\rangle$ have not to be isotropic in the XY plane. In the same way, the order-parameter propagators $G_i(\mathbf{q})=\langle\eta_i(\mathbf{q})\eta_i^*(\mathbf{q})\rangle$ are not isotropic, generally. The above properties of correlation functions are valid in the disordered hexatic phase as well, since the $N+6$ phase still preserves sixfold anisotropy.

For the sake of evidence, we explicitly show that, in spite of introducing an anisotropic gradient term that discriminates between directions in the XY plane, parallel and orthogonal to the sixfold crystallographic axes, respectively, the elastic energy of the lattice in the translationally ordered discotic phase remains isotropic. Moreover, as we shall see in Sec. IV, the renormalization of the model at one-loop order just produces such an anisotropic gradient term in the hexatic phase near the phase transition to the hexagonal discotic phase. Therefore, the anisotropy of the correlation functions does not contradict the elastic isotropy, since the renormalization procedure preserves the whole symmetry of the model [13].

Assuming an anisotropic gradient term in the plane of the lattice, the corresponding contribution to the free energy (throughout this paper, free energy is taken in terms of the k_BT unit) can be written as

$$\begin{aligned}F_{g\perp}=\tfrac{1}{2}\int d^3r\Big\{&M_{\perp1}^{-1}\sum_{i=1}^{3}|\mathbf{n}_i\cdot[\nabla_\perp-i\Omega_z(\mathbf{r})\mathbf{m}_0\times\mathbf{q}_i]\eta_i(\mathbf{r})|^2\\&+M_{\perp2}^{-1}\sum_{i=1}^{3}|\mathbf{n}_i\times[\nabla_\perp-i\Omega_z(\mathbf{r})\mathbf{m}_0\times\mathbf{q}_i]\\&\times\eta_i(\mathbf{r})|^2\Big\}\ ,\end{aligned} \tag{2.2}$$

where $\mathbf{n}_i=\mathbf{q}_i/q_0$ are the directions of $\mathbf{q}_i$ (q_0 being the modulus of $\mathbf{q}_i$, proportional to the inverse lattice spacing). With anisotropic coefficients $M_{\perp1}\neq M_{\perp2}$, the free energy (2.2) distinguishes between order-parameter fluctuations that develop longitudinally and transversally to the crystallographic axes, respectively. The isotropic counterpart of (2.2) is given in [4]. As a consequence of such an anisotropic structure of the gradient term, the order-parameter correlation function, in the hexatic phase near the transition temperature, is anisotropic. The order-parameter propagator is just (see also Sec. IV)

$$\begin{aligned}G_i(\mathbf{q})&\equiv\langle\eta_i(\mathbf{q})\eta_i^*(\mathbf{q})\rangle\\&=2[A+M_{\parallel}^{-1}q_z^2+M_{\perp1}^{-1}(\mathbf{n}_i\cdot\mathbf{q}_\perp)^2\\&\quad+M_{\perp2}^{-1}(\mathbf{n}_i\times\mathbf{q}_\perp)^2]^{-1}\ ,\end{aligned} \tag{2.3}$$

which should be compared with equation (39) of [4]. From Eq. (2.3), we can define anisotropic behavior for the correlation lengths, in the plane of the lattice too:

$$\xi_\parallel=(AM_\parallel)^{-1/2}\ , \tag{2.4a}$$

$$\xi_{\perp1}=(AM_{\perp1})^{-1/2}\ , \tag{2.4b}$$

$$\xi_{\perp2}=(AM_{\perp2})^{-1/2}\ , \tag{2.4c}$$

where $\xi_\parallel$, $\xi_{\perp1}$, and $\xi_{\perp2}$ are the correlation lengths in directions parallel to the direction of the liquid columns, parallel to the ith crystallographic axis, and orthogonal to both of them, respectively.

In the ordered discotic phase, we can assume a phase-

only order parameter [4]:

$$\eta_i(\mathbf{r})=\eta_0 e^{-i\mathbf{q}_i\cdot\mathbf{u}(\mathbf{r})} , \tag{2.5}$$

where η_0 is the constant amplitude and $\mathbf{u}(\mathbf{r})$ is the two-dimensional local displacement from the lattice sites. Substituting (2.5) in (2.2) and carrying out the sum over i, the elastic energy of the lattice can be obtained. In order to perform the sum over i, one has to exploit

$$\sum_{i=1}^{3} n_{i\alpha}n_{i\beta}=\tfrac{3}{2}\delta_{\alpha\beta} , \tag{2.6}$$

i.e., equation (6) of [4], and

$$\sum_{i=1}^{3} n_{i\alpha}n_{i\beta}n_{i\gamma}n_{i\delta}=\tfrac{3}{8}(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) , \tag{2.7}$$

with $\alpha,\beta,\ldots=x,y$. In fact, the hexagonal symmetry of the lattice makes the sum over the directions of the crystallographic axes isotropic, since the sixfold invariants are necessarily isotropic. As a consequence, one gets the isotropic elastic energy

$$F_{g\perp}=\tfrac{1}{2}\int d^3r(\lambda[\mathrm{div}\mathbf{u}(\mathbf{r})]^2+2\mu u_{\alpha\beta}(\mathbf{r})u_{\alpha\beta}(\mathbf{r}) +\tau\{\Omega_z(\mathbf{r})-\tfrac{1}{2}[\mathrm{rot}\mathbf{u}(\mathbf{r})]_z\}^2) , \tag{2.8}$$

where

$$u_{\alpha\beta}=\frac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right] \tag{2.9}$$

is the planar strain tensor of the lattice, with $\alpha,\beta=x,y$, and the elastic constants are

$$\lambda=\tfrac{3}{8}\eta_0^2q_0^2(M_{\perp 1}^{-1}-M_{\perp 2}^{-1}) , \tag{2.10}$$

$$\mu=\tfrac{3}{8}\eta_0^2q_0^2(M_{\perp 1}^{-1}+M_{\perp 2}^{-1}) , \tag{2.11}$$

$$\tau=3\eta_0^2q_0^2M_{\perp 2}^{-1} . \tag{2.12}$$

The ordinary Lamé coefficients [14] of two-dimensional elasticity are given in Eqs. (2.10) and (2.11), while τ in (2.12) is a torsion modulus, peculiar to the hexagonal discotic phase as a quasi-two-dimensional system. In fact, as one can see from (2.8), the elastic modulus τ represents the stiffness of the liquid columns against torsions around their equilibrium axes, and therefore points out that the structure of the system is not strictly two-dimensional. Each lattice site is just occupied by a columnar stack of disklike molecules with liquidlike behavior along the third dimension. Such a torsion elastic term is originated by the gauge coupling between the "rotation" field Ω_z and the order parameter, which just modifies the gradient term in the plane of the lattice, as discussed in [4].

The elastic energy (2.8) can also be written as

$$F_{g\perp}=\tfrac{1}{2}\int d^3r\{K(\mathrm{div}\mathbf{u})^2+2\mu(u_{\alpha\beta}-\tfrac{1}{2}\delta_{\alpha\beta}u_{\gamma\gamma})^2 +\tau[\Omega_z-\tfrac{1}{2}(\mathrm{rot}\mathbf{u})_z]^2\} , \tag{2.13}$$

with

$$K=\lambda+\mu=\tfrac{3}{4}\eta_0^2q_0^2M_{\perp 1}^{-1} . \tag{2.14}$$

In the form (2.13), the elastic energy of the lattice displays the three independent elastic modes of compression, shear and torsion, respectively. The elastic constant K, Eq. (2.14), and μ, Eq. (2.11), are just the two-dimensional compression and shear moduli, respectively. From Eq. (2.2), one can see that the order-parameter fluctuations along the crystallographic axes are controlled by the coefficient $M_{\perp 1}$ and then are a mixing of a compression mode and a shear mode, while the order-parameter fluctuations transversal to the crystallographic axes, being controlled by $M_{\perp 2}$, are a mixing of a torsion mode and a shear mode. In the particular case of an isotropic gradient, $M_{\perp 1}=M_{\perp 2}$, the Lamé coefficient λ, Eq. (2.10), vanishes and then the compression modulus K is equal to the shear modulus μ.

Considering also the gradient term along the liquid columns, whose coefficient is $M_{\parallel}^{-1}$ [4], we get the full elastic energy of the lattice:

$$F_g=\tfrac{1}{2}\int d^3r\{C(\nabla_z\mathbf{u}-\delta\mathbf{m})^2+K(\mathrm{div}\mathbf{u})^2 +2\mu(u_{\alpha\beta}-\tfrac{1}{2}\delta_{\alpha\beta}u_{\gamma\gamma})^2 +\tau[\Omega_z-\tfrac{1}{2}(\mathrm{rot}\mathbf{u})_z]^2\} , \tag{2.15}$$

where

$$C=\tfrac{3}{2}\eta_0^2q_0^2M_{\parallel}^{-1} \tag{2.16}$$

is a tilt modulus, representing the resistance of the molecules to be tilted with respect to the axis of the liquid column.

III. FRANK ENERGY AND THE GAUGE-FIELD PROPAGATORS

The full elastic energy associated with the rotation strains for discotic phases is [5,8]

$$F_{\mathrm{el}}=\tfrac{1}{2}\int d^3r\{K_1(\mathrm{div}\delta\mathbf{m})^2+K_2(\mathbf{m}_0\cdot\mathrm{rot}\delta\mathbf{m})^2 +K_3(\mathbf{m}_0\times\mathrm{rot}\delta\mathbf{m})^2+\gamma_1(\nabla_z\Omega_z)^2 +\gamma_2(\nabla_\perp\Omega_z)^2+2\gamma_3(\mathbf{m}_0\cdot\mathrm{rot}\delta\mathbf{m})(\nabla_z\Omega_z)\} , \tag{3.1}$$

where $\boldsymbol{\Omega}$ is the local rotation field defined in Eq. (1.1). As previously observed, the hexagonal symmetry implies cylindrical symmetry for the Frank energy. In particular, note that symmetry allows γ_3 coupling in free energy (3.1). As a consequence, $\delta\mathbf{m}$ and Ω_z are not independent fluctuating fields, since the twist mode of $\delta\mathbf{m}$ is coupled to the Ω_z "rotation" mode.

In Fourier space, the elastic energy (3.1) is

$$F_{\rm el}=\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\{(K_1q_x^2+K_2q_y^2+K_3q_z^2)|\delta m_x(\mathbf{q})|^2+(K_1q_y^2+K_2q_x^2+K_3q_z^2)|\delta m_y(\mathbf{q})|^2$$
$$+(K_1-K_2)q_xq_y[\delta m_x(\mathbf{q})\delta m_y^*(\mathbf{q})+\text{c.c.}]+(\gamma_1q_z^2+\gamma_2q_\perp^2)|\Omega_z(\mathbf{q})|^2$$
$$+\gamma_3q_zq_x[\delta m_y(\mathbf{q})\Omega_z^*(\mathbf{q})+\text{c.c.}]-\gamma_3q_zq_y[\delta m_x(\mathbf{q})\Omega_z^*(\mathbf{q})+\text{c.c.}]\}\ . \tag{3.2}$$

It is convenient to separate $\delta\mathbf{m}(\mathbf{q})$ into the superposition of the longitudinal and transverse components with respect to the projection of the wave vector $\mathbf{q}$ on the XY plane:

$$\delta\mathbf{m}(\mathbf{q})=\delta m_\perp(\mathbf{q})\mathbf{e}_\perp+\delta m_t(\mathbf{q})\mathbf{e}_t \tag{3.3}$$

where the longitudinal part

$$\delta m_\perp(\mathbf{q})=\frac{\mathbf{q}_\perp\cdot\delta\mathbf{m}(\mathbf{q})}{q_\perp}=\frac{q_x\delta m_x(\mathbf{q})+q_y\delta m_y(\mathbf{q})}{q_\perp} \tag{3.4}$$

is composed of splay and bend modes, while the transverse part

$$\delta m_t(\mathbf{q})=\frac{[\mathbf{q}_\perp\times\delta\mathbf{m}(\mathbf{q})]_z}{q_\perp}=\frac{q_x\delta m_y(\mathbf{q})-q_y\delta m_x(\mathbf{q})}{q_\perp} \tag{3.5}$$

is composed of twist and bend modes. The wave vector is

$$\mathbf{q}=q_z\mathbf{e}_z+q_\perp\mathbf{e}_\perp\ , \tag{3.6}$$

where $\mathbf{q}_\perp$ is just the projection of $\mathbf{q}$ on the XY plane and

$$\mathbf{e}_z\equiv\mathbf{m}_0\ , \tag{3.7a}$$

$$\mathbf{e}_\perp=\mathbf{q}_\perp/q_\perp\ , \tag{3.7b}$$

$$\mathbf{e}_t=\mathbf{e}_z\times\mathbf{e}_\perp \tag{3.7c}$$

is the orthonormal set. The elastic energy (3.2), in terms of the Fourier variables $\delta m_\perp$, δm_t, and Ω_z, is

$$F_{\rm el}=\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\{(K_1q_\perp^2+K_3q_z^2)|\delta m_\perp(\mathbf{q})|^2$$
$$+(K_2q_\perp^2+K_3q_z^2)|\delta m_t(\mathbf{q})|^2$$
$$+(\gamma_1q_z^2+\gamma_2q_\perp^2)|\Omega_z(\mathbf{q})|^2$$
$$+\gamma_3q_zq_\perp(\Omega_z(\mathbf{q})\delta m_t^*(\mathbf{q})+\text{c.c.})\}\ . \tag{3.8}$$

The application of the equipartition theorem to (3.8), which describes Gaussian field variable fluctuations, gives the field correlation functions in Fourier space. As one can see from (3.8), the longitudinal component $\delta m_\perp$ remains decoupled, while the transverse component δm_t is coupled to Ω_z. Therefore, the free (i.e., not coupled to the matter) propagators of the gauge fields are

$$D_\perp(\mathbf{q})\equiv\langle|\delta m_\perp(\mathbf{q})|^2\rangle=\frac{1}{K_1q_\perp^2+K_3q_z^2}\ , \tag{3.9a}$$

$$D_t(\mathbf{q})\equiv\langle|\delta m_t(\mathbf{q})|^2\rangle=\frac{\gamma_1q_z^2+\gamma_2q_\perp^2}{\Delta(\mathbf{q})}\ , \tag{3.9b}$$

$$T(\mathbf{q})\equiv\langle|\Omega_z(\mathbf{q})|^2\rangle=\frac{K_3q_z^2+K_2q_\perp^2}{\Delta(\mathbf{q})}\ , \tag{3.9c}$$

$$V(\mathbf{q})\equiv\langle\Omega_z(\mathbf{q})\delta m_t^*(\mathbf{q})\rangle=-\frac{\gamma_3q_zq_\perp}{\Delta(\mathbf{q})}\ , \tag{3.9d}$$

with

$$\Delta(\mathbf{q})=K_3\gamma_1q_z^4+K_2\gamma_2q_\perp^4+(K_3\gamma_2+K_2\gamma_1-\gamma_3^2)q_z^2q_\perp^2$$
$$=(K_3q_z^2+K_2q_\perp^2)(\gamma_1q_z^2+\gamma_2q_\perp^2)-\gamma_3^2q_z^2q_\perp^2\ . \tag{3.10}$$

For $\gamma_3\neq0$, the inverse propagators D_t^{-1} and T^{-1} are not quadratic in q. Since the elastic constant γ_3 is expected to have critical enhancement near the transition temperature [8], the fields δm_t and Ω_z remain strongly coupled.

For further applications, it is convenient to factorize the denominator $\Delta(\mathbf{q})$, Eq. (3.10), as

$$\Delta(\mathbf{q})=K_3\gamma_1[q_z^2+(a+b)q_\perp^2][q_z^2+(a-b)q_\perp^2]\ , \tag{3.11}$$

with

$$a=\frac{K_3\gamma_2+K_2\gamma_1-\gamma_3^2}{2K_3\gamma_1},\quad b=\left[a^2-\frac{K_2\gamma_2}{K_3\gamma_1}\right]^{1/2}\ , \tag{3.12}$$

so that the propagators (3.9b), (3.9c), and (3.9d), can be rearranged in the form

$$D_t(\mathbf{q})=\frac{1}{2K_3b}\left[\frac{b-a+c}{q_z^2+(a+b)q_\perp^2}+\frac{b+a-c}{q_z^2+(a-b)q_\perp^2}\right]\ , \tag{3.13a}$$

$$T(\mathbf{q})=\frac{1}{2\gamma_2b}\left[\frac{(a+b)(b+a-c)}{q_z^2+(a+b)q_\perp^2}+\frac{(a-b)(b-a+c)}{q_z^2+(a-b)q_\perp^2}\right]\ , \tag{3.13b}$$

$$V(\mathbf{q})=-\frac{\gamma_3}{K_3\gamma_1}\frac{q_zq_\perp}{[q_z^2+(a+b)q_\perp^2][q_z^2+(a-b)q_\perp^2]}\ , \tag{3.13c}$$

with

$$c=\frac{K_2\gamma_1-\gamma_3^2}{K_3\gamma_1}\ . \tag{3.14}$$

A similar splitting was employed in [15] to treat disclination strains and pair-interaction energy. As regards angular dependence of propagators (3.13), letting

$$q_z=q\cos\theta,\quad q_\perp=q\sin\theta\ , \tag{3.15}$$

where θ is the angle between $\mathbf{q}$ and $\mathbf{m}_0$, and defining

$$D_\perp(\mathbf{q})\equiv\hat{D}_\perp(\theta)/q^2\ , \tag{3.16a}$$

$$D_t(\mathbf{q})\equiv\hat{D}_t(\theta)/q^2\ , \tag{3.16b}$$

$$T(\mathbf{q})\equiv\hat{T}(\theta)/q^2\ , \tag{3.16c}$$

$$V(\mathbf{q})\equiv\hat{V}(\theta)/q^2\ , \tag{3.16d}$$

we have

$$\hat{D}_\perp = \frac{1}{K_1 \sin^2\theta + K_3 \cos^2\theta} , \tag{3.17a}$$

$$\hat{D}_t = \frac{1}{2K_3 b}\left[\frac{b-a+c}{\cos^2\theta+(a+b)\sin^2\theta}+\frac{b+a-c}{\cos^2\theta+(a-b)\sin^2\theta}\right] , \tag{3.17b}$$

$$\hat{T} = \frac{1}{2\gamma_2 b}\left[\frac{(a+b)(b+a-c)}{\cos^2\theta+(a+b)\sin^2\theta}+\frac{(a-b)(b-a+c)}{\cos^2\theta+(a-b)\sin^2\theta}\right] , \tag{3.17c}$$

$$\hat{V} = -\frac{\gamma_3}{K_3\gamma_1}\frac{\cos\theta\sin\theta}{[\cos^2\theta+(a+b)\sin^2\theta][\cos^2\theta+(a-b)\sin^2\theta]} , \tag{3.17d}$$

while they are independent of the azimuthal angle φ in the XY plane.

The elastic energy (3.8) can be settled in close form, by defining the strain vector $\boldsymbol{\nu}$, the components of which are $\nu_x \equiv \delta m_x$, $\nu_y \equiv \delta m_y$, and $\nu_z \equiv \Omega_z$. The Frank energy is then

$$F_{\rm el} = \tfrac{1}{2}\int \frac{d^3q}{(2\pi)^3} Q_{\alpha\beta}(\mathbf{q})\nu_\alpha(\mathbf{q})\nu_\beta^*(\mathbf{q}) , \tag{3.18}$$

where $Q_{\alpha\beta}$ is the inverse, in the matrix sense, of the correlation function tensor

$$P_{\alpha\beta}(\mathbf{q}) = \langle \nu_\alpha(\mathbf{q})\nu_\beta^*(\mathbf{q})\rangle . \tag{3.19}$$

The tensorial structure of the gauge-field propagator is thus

$$P_{\alpha\beta} = D_{\alpha\beta} + T\delta_{z\alpha}\delta_{z\beta} + V(\delta_{z\alpha}e_{t\beta}+\delta_{z\beta}e_{t\alpha}) , \tag{3.20}$$

with

$$D_{\alpha\beta} = D_\perp e_{\perp\alpha}e_{\perp\beta} + D_t e_{t\alpha}e_{t\beta} , \tag{3.21}$$

where $D_\perp$, D_t, T, and V are given in Eqs.(3.9) or (3.13), and the orthonormal triad is in (3.7).

IV. RENORMALIZATION AND ANISOTROPIC SCALING

The full free energy for a discotic liquid crystal is [4,5,8]

$$F = F_0 + F_{\rm el} , \tag{4.1}$$

where $F_{\rm el}$ is given in Eq. (3.1) and F_0, by an appropriate rescaling of lengths and fields [10], can be written as

$$F_0 = \tfrac{1}{2}\int d^3r \left\{ A\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^2 + \frac{B}{\sqrt{2}}\eta_1(\mathbf{r})\eta_2(\mathbf{r})\eta_3(\mathbf{r}) + \frac{1}{2}C_1\left[\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^2\right]^2 + \frac{1}{2}C_2\sum_{i=1}^{3}|\eta_i(\mathbf{r})|^4 + \sum_{i=1}^{3}\left(|[\nabla_z + i\mathbf{q}_i\cdot\delta\mathbf{m}(\mathbf{r})]\eta_i(\mathbf{r})|^2 + |[\nabla_\perp - i\Omega_z(\mathbf{r})\mathbf{m}_0\times\mathbf{q}_i]\eta_i(\mathbf{r})|^2\right)\right\} , \tag{4.2}$$

with, as usual in Landau free energy, $A = A_0(T-T^*)$, T^* being a second-order phase-transition temperature, while the other coefficients are positive constants. According to the Landau criterion [6], and then in the mean-field approximation, for small coefficient B, the transition is weakly first order [4]. This free energy is composed of two contributions. The Frank energy $F_{\rm el}$ is the energy of the pure gauge field $\boldsymbol{\Omega}$, Eq. (1.1), and furnishes, as seen in Sec. III, the propagators of the free gauge fields, not coupled to the matter, represented by the density-wave parameters η_i, Eq. (2.1). The free energy F_0 contains the self-energy and the self-interaction of the order-parameter fields η_i, as well as the gauge coupling between η_i and $\boldsymbol{\Omega}$. The gradient term in (4.2) has been made isotropic by the suitable rescaling of lengths, in order to get an isotropic bare propagator for the order parameter [10]. In this way, the anisotropy of the renormalized propagator is entirely due to the gauge interaction.

Since our aim is to investigate anisotropic critical properties of the order-parameter propagator, we focus our attention on gauge coupling. The vertex interactions originated by gauge coupling are represented in Fig. 1. In fact, anisotropic behavior is yielded by the tensorial structures of such vertex interactions, which distinguish between directions parallel and orthogonal to the shortest reciprocal lattice vectors $\mathbf{q}_i$, respectively. Actually, as we shall see in the following, only the three-point vertices [Figs. 1(a) and 1(c)] are effective in giving anisotropic contributions to the order-parameter propagator, at one-loop order. On the contrary, the order-parameter self-interactions, as the cubic and quartic terms in (4.2), cannot lead to anisotropic contributions.

The order-parameter propagator $G_i(\mathbf{q}) = \langle \eta_i(\mathbf{q})\eta_i^*(\mathbf{q})\rangle$ is renormalized at one-loop order by the self-interactions of the density parameter η_i and by the gauge interactions represented in Fig. 1. As a consequence, the inverse order-parameter propagator at one-loop order is given by

$$G^{-1}(\mathbf{q}) = G_0^{-1}(\mathbf{q}) + \Gamma(\mathbf{q}) , \tag{4.3}$$

where the index i has been dropped for compactness. In (4.3), $G_0(\mathbf{q})$ is the free propagator of the order parameter, i.e., the propagator in the absence of gauge coupling and self-coupling, while $\Gamma(\mathbf{q})$ is the self-energy due to one-loop corrections. From (4.2), in Gaussian approximation and for an isotropic bare gradient, the free propagator is then

$$G_0(\mathbf{q})=\frac{2}{A+q^2} . \tag{4.4}$$

The renormalization-group transformation consists of two steps. Step one is the integration over short-wavelength field fluctuations, which yields the perturbative contributions to the vertex functions and to the propagators, e.g., the correction $\Gamma(\mathbf{q})$ to $G_0^{-1}(\mathbf{q})$ in (4.3). Step two is the rescaling of lengths, in order to recover the primitive length scale of fluctuations. In such a way, at the end of the recurrent procedure, one obtains the primitive free energy in terms of the renormalized parameters (see, e.g., [13]).

The contributions to $\Gamma(\mathbf{q})$ at the one-loop level are given in Fig. 2. The diagrams that involve four-point vertices [Figs. 2(a)–2(c)] are independent of the external wave vector $\mathbf{q}$ and therefore only renormalize the parameter A in free energy (4.2), i.e., the second-order phase-transition temperature T^*. On the contrary, the diagrams in Figs. 2(d)–2(g), built up with three-point vertices, depend on $\mathbf{q}$, and then also renormalize the gra-

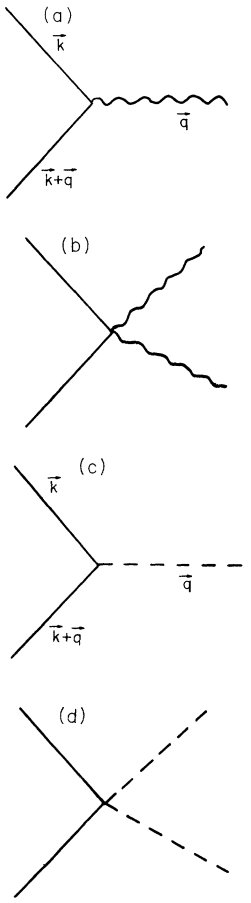

FIG. 1. Vertex interactions due to gauge coupling. The wiggly line represents the gauge field $\delta\mathbf{m}$, while the dashed line represents the gauge field Ω_z. The solid line stands for the order-parameter field η_i. The three-point vertex (a) is $\frac{1}{2}q_{i\alpha}(2K_z+q_z)$, while the four-point vertex (b) is $\frac{1}{2}q_{i\alpha}q_{i\beta}$. The three-point vertex (c) is equal to $-\frac{1}{2}(\mathbf{m}_0\times\mathbf{q}_i)_\alpha(2K_{\perp\alpha}+q_{\perp\alpha})$, and the four-point vertex (d) is $\frac{1}{2}q_0^2$. In (a) and (c) the wave vectors carried by the lines are displayed.

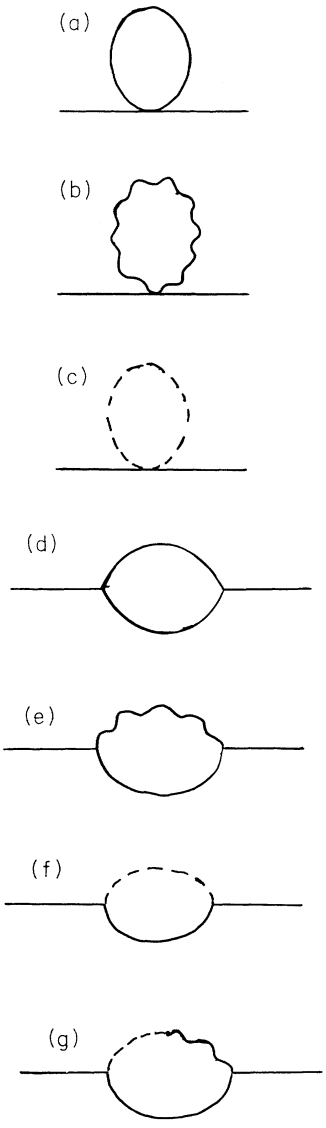

FIG. 2. One-loop diagram contributing to $\Gamma(\mathbf{q})$. The wiggly line represents $D_{\alpha\beta}(\mathbf{q})$, the dashed line represents $T(\mathbf{q})$, and the mixed wiggly-dashed line stands for $V(\mathbf{q})$, which are defined in Sec. III as the gauge-field propagators. The solid line represents $G(\mathbf{q})$, i.e., the propagator of the order parameter. The diagrams (a)–(c), which involve four-point interactions, are independent of the external wave vector $\mathbf{q}$. On the contrary, the diagrams (d)–(g), which are built up with three-point vertices, depend on $\mathbf{q}$ and then renormalize the gradient coefficients.

dient coefficients of (4.2). Figure 2(d), due to cubic self-interaction, gives isotropic renormalization. Figures 2(e)–2(g), because of the tensorial structure of gauge interaction vertices (see Fig. 1), produce anisotropic contributions to the gradient coefficients. Therefore, each iteration of the renormalization procedure gives rise to anisotropic self-energy $\Gamma(\mathbf{q})$. Anisotropic rescaling is then necessary, in order to regain the original isotropy of the propagator (4.4) (see also [10]). In that way, the renormalized propagator $G_i(\mathbf{q})$ should fulfill the homogeneity relation [10]

$$G_i(\mathbf{q},t) = \Lambda^{2-\eta_\parallel} G_i(\Lambda q_z, \Lambda^{1+\mu_{\perp 1}}|\mathbf{q}_\perp\cdot\mathbf{n}_i|, \Lambda^{1+\mu_{\perp 2}}|\mathbf{q}_\perp\times\mathbf{n}_i|, \Lambda^{1/\nu_\parallel} t) , \tag{4.5}$$

where $t=(T-T^*)/T^*$, while $|\mathbf{q}_\perp\cdot\mathbf{n}_i|$ and $|\mathbf{q}_\perp\times\mathbf{n}_i|$ are the longitudinal and transversal components of $\mathbf{q}_\perp$ with respect to the directions $\mathbf{n}_i$ of the crystallographic axes, respectively. Anisotropic scaling of correlation lengths follows from (4.5):

$$\xi_\parallel \sim |t|^{-\nu_\parallel} , \tag{4.6a}$$

$$\xi_{\perp 1} \sim |t|^{-\nu_{\perp 1}} , \tag{4.6b}$$

$$\xi_{\perp 2} \sim |t|^{-\nu_{\perp 2}} , \tag{4.6c}$$

with

$$\nu_{\perp 1} = (1+\mu_{\perp 1})\nu_\parallel , \tag{4.7a}$$

$$\nu_{\perp 2} = (1+\mu_{\perp 2})\nu_\parallel , \tag{4.7b}$$

where $\mu_{\perp 1}$ and $\mu_{\perp 2}$ are the anisotropic scaling indices, while $\nu_\parallel$, $\nu_{\perp 1}$, and $\nu_{\perp 2}$ are the anisotropic exponents with which correlation lengths along different directions diverge for $T\to T^*$.

The anisotropic contribution to $\Gamma(\mathbf{K})$ due to the diagram in Fig. 2(e) is

$$\Gamma_1(\mathbf{K}) = -\sum_{\alpha,\beta} q_{i\alpha} q_{i\beta} \int_{1\ge q\ge\Lambda^{-1}} \frac{d^3q}{(2\pi)^3} G_0(\mathbf{K}+\mathbf{q}) D_{\alpha\beta}(\mathbf{q}) \times (2K_z+q_z)^2 , \tag{4.8}$$

where $d^3q = q^2 dq\, d\varphi\, \sin\theta\, d\theta$. The gauge-field propagator $D_{\alpha\beta}$ is defined in (3.21), and G_0 in (4.4). The integration in (4.8) is made over the shell $1\ge|\mathbf{q}|\ge\Lambda^{-1}$, which removes short wave lengths at each iteration. This process creates anisotropy in the coefficients of the term quadratic in K. Developing the integrand expression of (4.8) up to second order in K and performing the angular integration over the azimuthal angle φ in the XY plane, we get

$$\begin{aligned}\Gamma_1(\mathbf{K}) = -\left[\frac{q_0}{2\pi}\right]^2 \{ & \overline{(\hat{D}_\perp+\hat{D}_t)\cos^2\theta}\, I_0(\Lambda) \\ & + K_z^2[4\overline{(\hat{D}_\perp+\hat{D}_t)} I_1(\Lambda) - \tfrac{9}{2}\overline{(\hat{D}_\perp+\hat{D}_t)\cos^2\theta}\, I_2(\Lambda) + \overline{(\hat{D}_\perp+\hat{D}_t)\cos^4\theta}\, I_3(\Lambda)] \\ & + (\mathbf{n}_i\cdot\mathbf{K}_\perp)^2[-\tfrac{1}{2}\overline{(\hat{D}_\perp+\hat{D}_t)\cos^2\theta}\, I_2(\Lambda) + \tfrac{1}{4}\overline{(3\hat{D}_\perp+\hat{D}_t)\cos^2\theta\sin^2\theta}\, I_3(\Lambda)] \\ & + (\mathbf{n}_i\times\mathbf{K}_\perp)^2[-\tfrac{1}{2}\overline{(\hat{D}_\perp+\hat{D}_t)\cos^2\theta}\, I_2(\Lambda) + \tfrac{1}{4}\overline{(\hat{D}_\perp+3\hat{D}_t)\cos^2\theta\sin^2\theta}\, I_3(\Lambda)]\} ,\end{aligned} \tag{4.9}$$

where we have defined

$$I_0(\Lambda) \equiv \int_{\Lambda^{-1}}^1 q^2 G_0(q) dq , \tag{4.10a}$$

$$I_1(\Lambda) \equiv \int_{\Lambda^{-1}}^1 G_0(q) dq , \tag{4.10b}$$

$$I_2(\Lambda) \equiv \int_{\Lambda^{-1}}^1 q^2 G_0^2(q) dq , \tag{4.10c}$$

$$I_3(\Lambda) \equiv \int_{\Lambda^{-1}}^1 q^4 G_0^3(q) dq , \tag{4.10d}$$

while $\hat{D}_\perp$ and $\hat{D}_t$ are given in (3.17a) and (3.17b), respectively, and the angular average over θ is defined as

$$\bar{f} \equiv \tfrac{1}{2}\int_0^\pi f(\theta)\sin\theta\, d\theta . \tag{4.11}$$

The various angular averages are given in the Appendix.

The anisotropic contribution to $\Gamma(\mathbf{K})$, which comes from Fig. 2(f), is

$$\Gamma_2(\mathbf{K}) = -\sum_{\alpha,\beta} (\mathbf{m}_0\times\mathbf{q}_i)_\alpha (\mathbf{m}_0\times\mathbf{q}_i)_\beta \times \int_{1\ge q\ge\Lambda^{-1}} \frac{d^3q}{(2\pi)^3} G_0(\mathbf{K}+\mathbf{q}) T(\mathbf{q}) \times (2K_{\perp\alpha}+q_{\perp\alpha})(2K_{\perp\beta}+q_{\perp\beta}) , \tag{4.12}$$

where $T(\mathbf{q})$ is the propagator of Ω_z, Eq. (3.13b). As for the previous contribution, at the second order in K, we obtain anisotropic gradient coefficients

$$\begin{aligned}\Gamma_2(\mathbf{K}) = -\left[\frac{q_0}{2\pi}\right]^2 \{ & \overline{\hat{T}\sin^2\theta}\, I_0(\Lambda) + K_z^2[\overline{\hat{T}\sin^2\theta\cos^2\theta}\, I_3(\Lambda) - \tfrac{1}{2}\overline{\hat{T}\sin^2\theta}\, I_2(\Lambda)] \\ & + (\mathbf{n}_i\cdot\mathbf{K}_\perp)^2[\tfrac{1}{4}\overline{\hat{T}\sin^4\theta}\, I_3(\Lambda) - \tfrac{1}{2}\overline{\hat{T}\sin^2\theta}\, I_2(\Lambda)] \\ & + (\mathbf{n}_i\times\mathbf{K}_\perp)^2[8\bar{\hat{T}} I_1(\Lambda) - \tfrac{9}{2}\overline{\hat{T}\sin^2\theta}\, I_2(\Lambda) + \tfrac{3}{4}\overline{\hat{T}\sin^4\theta}\, I_3(\Lambda)]\} ,\end{aligned} \tag{4.13}$$

where $\hat{T}$ is given in (3.17c).

The contribution of Fig. 2(g) is

$$\Gamma_3(\mathbf{K})=2\sum_{\alpha,\beta}(\mathbf{m}_0\times\mathbf{q}_i)_\alpha q_{i\beta}\int_{1\geq q\geq\Lambda^{-1}}\frac{d^3q}{(2\pi)^3}G_0(\mathbf{K}+\mathbf{q})V(\mathbf{q})e_{t\beta}(2K_z+q_z)(2K_{\perp\alpha}+q_{\perp\alpha}) , \tag{4.14}$$

where $V(\mathbf{q})$ is the mixing between Ω_z and δm_t, given in Eqs. (3.9d) or (3.13c). Equation (4.14) at second order in K gives

$$\begin{aligned}\Gamma_3(\mathbf{K})=-\left[\frac{q_0}{2\pi}\right]^2\{&2\overline{\hat{V}\sin\theta\cos\theta}\,I_0(\Lambda)+K_z^2[-5\overline{\hat{V}\sin\theta\cos\theta}\,I_2(\Lambda)+2\overline{\hat{V}\sin\theta\cos^3\theta}\,I_3(\Lambda)]\\&+(\mathbf{n}_i\cdot\mathbf{K}_\perp)^2[-\overline{\hat{V}\sin\theta\cos\theta}\,I_2(\Lambda)+\tfrac{1}{2}\overline{\hat{V}\sin^3\theta\cos\theta}\,I_3(\Lambda)]\\&+(\mathbf{n}_i\times\mathbf{K}_\perp)^2[-5\overline{\hat{V}\sin\theta\cos\theta}\,I_2(\Lambda)+\tfrac{3}{2}\overline{\hat{V}\sin^3\theta\cos\theta}\,I_3(\Lambda)]\} ,\end{aligned} \tag{4.15}$$

where $\hat{V}$ is given in (3.17d).

Collecting the anisotropic contributions to the self-energy, Eqs. (4.9), (4.13), and (4.15), we obtain the propagator $G_i(\mathbf{q})$, renormalized at the one-loop level, of the form (2.3). In particular, we emphasize the anisotropy of $G_i(\mathbf{q})$ in the XY plane with respect to the $\mathbf{n}_i$ crystallographic axis, whose vestige remains in the hexatic phase near the phase transition to the hexagonal discotic phase. Such anisotropy follows from the tensorial structure of the gauge coupling in our model [4], and can be expressed by the difference between $M_{\perp1}^{-1}$ and $M_{\perp2}^{-1}$ of (2.3) which, at the one-loop level, is

$$\begin{aligned}M_{\perp1}^{-1}-M_{\perp2}^{-1}=4\left[\frac{q_0}{2\pi}\right]^2\{&4\overline{\hat{T}}I_1(\Lambda)-2(\overline{\hat{T}\sin^2\theta}+\overline{\hat{V}\sin\theta\cos\theta})I_2(\Lambda)\\&+\tfrac{1}{4}[(\overline{\hat{D}_t-\hat{D}_\perp})\cos^2\theta\sin^2\theta+\overline{\hat{T}\sin^4\theta}+2\overline{\hat{V}\sin^3\theta\cos\theta}]I_3(\Lambda)\} .\end{aligned} \tag{4.16}$$

The result (4.16) depends on the scale Λ of renormalization, and therefore acquires physical sense only when, after many recurrences, the renormalization-group transformation brings the system towards the fixed point corresponding to the critical region of the phase transition. The anisotropy (4.16), if it is not vanishing at the critical fixed point, then yields anisotropic scaling indices $\mu_{\perp1}\neq\mu_{\perp2}$, as defined in (4.7).

V. DISCUSSION

We have studied renormalization corrections, at one-loop order, of our model [4,5] for the transition of the hexatic $N+6$ phase into the hexagonal discotic phase. As a result, the gauge coupling between the order parameter and the curvature field $\boldsymbol{\Omega}$ drives critical behavior of correlation lengths, which can be experimentally investigated from the structure factor for x-ray diffraction. In particular, the tensorial structure of the gauge coupling yields anisotropic corrections to the order-parameter correlation function, even in the plane orthogonal to the nematic director with respect to the directions of the sixfold crystallographic axes.

Such anisotropy of the order-parameter correlation function might determine anisotropic scaling indices for the correlation lengths along different directions. Actually, one should compute the anisotropy at the critical fixed point of renormalization flow, where the physical quantities become independent of the scale of renormalization. The occurrence of anisotropic scaling therefore depends on the existence of such an anisotropic fixed point and on its stability. The inverse propagators of the gauge fields are not quadratic in the wave vector, owing to the mixing between the director elastic modes and the "rotation" elastic modes. Because of some complications about cumbersome calculations related to such a peculiar structure of the gauge-field propagators, we have not been able to find and analyze the fixed point controlling the phase transition.

Some analogies between our model [4,8] and the De Gennes model for the nematic-to-smectic-A transition [9] would suggest a series of crossovers, from anisotropic critical behavior to isotropic critical behavior and finally to a first-order transition, as for smectic phases [10]. However, this result for smectic phases is based on ϵ expansion in the vicinity of four dimensions and therefore could not be simply extended to our model. In fact, it is not clear how to generalize our model in arbitrary D dimensions. For the physical value $D=3$, the ordered phase can be considered solid in two dimensions and liquid in the third dimension [4]. In arbitrary D dimensions, we could generalize the model assuming solidlike behavior in $D-1$ dimensions and liquidlike behavior in the remaining dimension. But, in that way, we should deal with an order parameter whose number of components depends on the number of space dimensions, since we ought to describe the onset of density waves in arbitrary $D-1$ dimensions. Otherwise, we could assume two "solid" dimensions and $D-2$ "liquid" dimensions, which would modify the Frank energy introducing new

elastic constants. Moreover, the particular gauge coupling of our model, as in Eq. (4.2), is strictly related to three dimensions, and it is not clear what role the unphysical dimensions should play. Anyhow, the structure of the model would change with the number of dimensions.

Therefore, the framework of dimensional regularization should not be suitable to our model. On the contrary, we should use the fixed-dimension formalism developed by Parisi [16], in $D=3$. That is why, in Sec. IV, we compute renormalization corrections in $D=3$. At present, in the absence of an understanding of the fixed point relative to the transition between the hexatic $N+6$ phase and the hexagonal discotic phase, we cannot state anisotropic scaling but only conclude about its possibility, based essentially on symmetry grounds.

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APPENDIX

In this appendix, we give some angular averages that enter the self-energy and then determine the anisotropy (4.16) between the renormalized gradient coefficients. Such averages of propagators of the gauge fields express the effect of orientational order on the order-parameter propagator. The average over θ, the angle comprised between the wave vector $\mathbf{q}$ and the unperturbed nematic director $\mathbf{m}_0$, is defined in (4.11).

All averages can be expressed in terms of the general integral

$$L_{nm}(\alpha,\beta)=\tfrac{1}{2}\int_0^{\pi}\frac{\sin^{2n}\theta\cos^{2m}\theta}{\alpha\sin^2\theta+\beta\cos^2\theta}\sin\theta\,d\theta\ , \tag{A1}$$

which can be evaluated using

$$\begin{aligned}L_{00}(\alpha,\beta)&=\tfrac{1}{2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{\alpha\sin^2\theta+\beta\cos^2\theta}\\&=\left[\frac{1}{\alpha(\beta-\alpha)}\right]^{1/2}\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2},\end{aligned} \tag{A2}$$

and some simple reduction formulas. The integrals we need, besides (A2), are

$$L_{01}(\alpha,\beta)=\frac{1}{\beta-\alpha}\left[1-\left[\frac{\alpha}{\beta-\alpha}\right]^{1/2}\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2}\right], \tag{A3}$$

$$L_{10}(\alpha,\beta)=\frac{1}{\beta-\alpha}\left[-1+\frac{\beta}{\alpha^{1/2}(\beta-\alpha)^{1/2}}\times\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2}\right], \tag{A4}$$

$$L_{02}(\alpha,\beta)=\frac{1}{3(\beta-\alpha)}-\frac{\alpha}{(\beta-\alpha)^2}\left[1-\left[\frac{\alpha}{\beta-\alpha}\right]^{1/2}\times\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2}\right], \tag{A5}$$

$$L_{11}(\alpha,\beta)=\frac{2}{3(\beta-\alpha)}+\frac{\alpha}{(\beta-\alpha)^2}\left[1-\frac{\beta}{\alpha^{1/2}(\beta-\alpha)^{1/2}}\times\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2}\right], \tag{A6}$$

$$L_{20}(\alpha,\beta)=-\frac{5}{3(\beta-\alpha)}+\frac{\alpha}{(\beta-\alpha)^2}\left[-1+\frac{\beta^2}{\alpha^{3/2}(\beta-\alpha)^{1/2}}\times\arctan\left[\frac{\beta-\alpha}{\alpha}\right]^{1/2}\right]. \tag{A7}$$

Taking into account the definitions of $\hat{D}_\perp$, $\hat{D}_t$, $\hat{T}$, and $\hat{V}$ in (3.17), we obtain the angular averages, in terms of (A2)–(A7), as

$$\bar{\hat{D}}_\perp=L_{00}(K_1,K_3)\ , \tag{A8}$$

$$\bar{\hat{D}}_t=\frac{1}{2b}[(b-a+c)L_{00}(K_3(a+b),K_3)+(b+a-c)L_{00}(K_3(a-b),K_3)]\ , \tag{A9}$$

$$\bar{\hat{T}}=\frac{1}{2b}[(a+b)(b+a-c)L_{00}(\gamma_2(a+b),\gamma_2)+(a-b)(b-a+c)L_{00}(\gamma_2(a-b),\gamma_2)]\ , \tag{A10}$$

$$\overline{\hat{D}_\perp\cos^2\theta}=L_{01}(K_1,K_3)\ , \tag{A11}$$

$$\overline{\hat{D}_t\cos^2\theta}=\frac{1}{2b}[(b-a+c)L_{01}(K_3(a+b),K_3)+(b+a-c)L_{01}(K_3(a-b),K_3)]\ , \tag{A12}$$

$$\overline{\hat{T}\sin^2\theta}=\frac{1}{2b}[(a+b)(b+a-c)L_{10}(\gamma_2(a+b),\gamma_2)+(a-b)(b-a+c)L_{10}(\gamma_2(a-b),\gamma_2)]\ , \tag{A13}$$

$$\overline{\hat{V}\sin\theta\cos\theta}=\frac{\gamma_3}{2b\gamma_1}[L_{01}(K_3(a+b),K_3)-L_{01}(K_3(a-b),K_3)]\ , \tag{A14}$$

$$\overline{\hat{D}_\perp\cos^4\theta}=L_{02}(K_1,K_3)\ , \tag{A15}$$

$$\overline{\hat{D}_t\cos^4\theta}=\frac{1}{2b}[(b-a+c)L_{02}(K_3(a+b),K_3)+(b+a-c)L_{02}(K_3(a-b),K_3)]\ , \tag{A16}$$

$$\overline{\hat{T}\sin^4\theta}=\frac{1}{2b}[(a+b)(b+a-c)L_{20}(\gamma_2(a+b),\gamma_2)+(a-b)(b-a+c)L_{20}(\gamma_2(a-b),\gamma_2)]\ , \tag{A17}$$

$$\overline{\hat{D}_\perp\sin^2\theta\cos^2\theta}=L_{11}(K_1,K_3)\ , \tag{A18}$$

$$\overline{\hat{D}_t\sin^2\theta\cos^2\theta}=\frac{1}{2b}[(b-a+c)L_{11}(K_3(a+b),K_3)+(b+a-c)L_{11}(K_3(a-b),K_3)]\ , \tag{A19}$$

$$\overline{\hat{T}\sin^2\theta\cos^2\theta}=\frac{1}{2b}[(a+b)(b+a-c)L_{11}(\gamma_2(a+b),\gamma_2)+(a-b)(b-a+c)\times L_{11}(\gamma_2(a-b),\gamma_2)]\ , \tag{A20}$$

$$\overline{\hat{V}\sin^3\theta\cos\theta}=\frac{\gamma_3}{2b\gamma_1}[L_{11}(K_3(a+b),K_3)-L_{11}(K_3(a-b),K_3)]\ , \tag{A21}$$

$$\overline{\hat{V}\sin\theta\cos^3\theta}=\frac{\gamma_3}{2b\gamma_1}[L_{02}(K_3(a+b),K_3)-L_{02}(K_3(a-b),K_3)]\ . \tag{A22}$$

Note that the angular averages entering the anisotropy (4.16) are only (A10), (A13), (A14), (A17), (A18), (A19), and (A21).

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